Holographic flow of Anomalous Transport Coefficients

Work in Progress

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Issues

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QFT in the hydrodynamic regime

Global equilibrium
Ideal fluid

$$T_{\mu\nu} = (P + \epsilon) u_\mu u_\nu + P g_{\mu\nu} \tag{1}$$

Dual theory: Black Hole

Grand canonical ensemble: Global symmetries of the CFT

$$J^\mu = n u^\mu \tag{2}$$

Dual theory: Gauge fields propagating in the bulk

$$\mu \sim A_0(B) - A_0(H) \tag{3}$$

(T, Q_I) of the CFT $\rightarrow$ (T_H, Q_I) of the B-H

Perturbing the peace

- Perturbations with small amplitude.

  Linear response theory: Kubo formulas → transport coefficients.

- Perturbations with small energy (late-time)

  First order Hydro

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + \Pi^{\mu\nu} \tag{4}$$

$$J^{\mu} = n u^{\mu} + \nu^{\mu} \tag{5}$$

Fluid/gravity correspondence → transport coefficients.

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  First order Hydro + Kubo formulas applicable!

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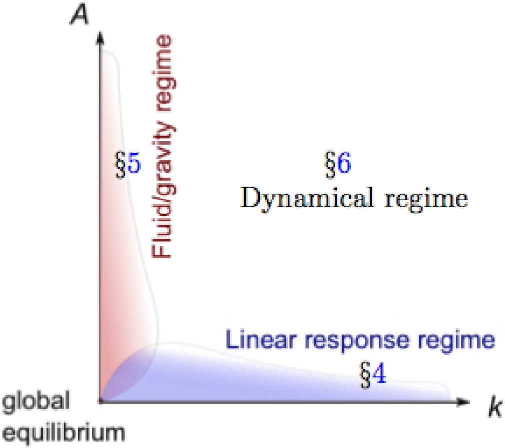
  First order Hydro + Kubo formulas applicable!

Perturbing the peace II ([Hubeny, Rangamani '10])



Anomalies: Generalities

- Axial anomaly [Bell, Jackiw '69]

$$D_\mu j_5^\mu \propto \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{6}$$

- Mixed anomaly [Delbourgo, Salam '72]

$$D_\mu j_5^\mu \propto \epsilon^{\mu\nu\rho\sigma} R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\sigma} \tag{7}$$

- Both can be implemented by including Chern-Simons terms in the bulk action, i.e. [Son, Surówka '09]

$$\Delta S \sim \int d^5x \sqrt{-g} \left(\frac{4\kappa}{3} \epsilon^{LABCD} A_L F_{AB} F_{CD} \right) \tag{8}$$

Anomalous transport I

- Chiral Magnetic Effect: Appearance of a current in the direction of $\vec{B}$ [Kharzeev, Warringa '09]

$$\delta J^\mu = \sigma_B \epsilon^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma} \tag{9}$$

$$\delta T^{\mu\nu} = \sigma_B^\epsilon (u^\mu B^\nu + (\mu \to \nu)) \tag{10}$$

$$\sigma_B = \lim_{k_c \to 0} \frac{i}{2k_c} \sum_{a,b} \epsilon_{abc} \left\langle J^a J^b \right\rangle (\omega = 0, \vec{p}) \tag{11}$$

$$\sigma_B^\epsilon = \lim_{k_c \to 0} \frac{i}{2k_c} \sum_{a,b} \epsilon_{abc} \left\langle T^{0a} J^b \right\rangle (\omega = 0, \vec{p}) \tag{12}$$

where $a, b, c = x, y, z$.

Anomalous transport II

- Chiral Vortical Effect: Appearance of a current due to vortices in the fluid ω^m [Bhattacharyya, Hubeny, Minwalla, Rangamani '07].

$$\delta J^\mu = \sigma_V \epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma \tag{13}$$

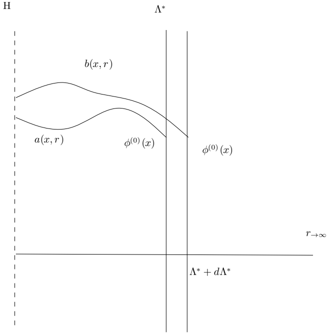
$$\delta T^{\mu\nu} = \sigma_V^\epsilon (u^\mu \omega^\nu + (\mu \to \nu)) \tag{14}$$

$$\sigma_V = \lim_{k_c \to 0} \frac{i}{2k_c} \sum_{a,b} \epsilon_{abc} \left\langle J^a T^{0b} \right\rangle (\omega = 0, A_0 = 0) \tag{15}$$

$$\sigma_V^\epsilon = \lim_{k_c \to 0} \frac{i}{2k_c} \sum_{a,b} \epsilon_{abc} \left\langle T^{0a} T^{0b} \right\rangle (\omega = 0, \vec{p}) \tag{16}$$

where $a, b, c = x, y, z$.

Cutoff flow [Iqbal, Liu '08] [Faulkner, Liu, Rangamani '11]



Two theories, one equipped with a cutoff Λ^* and the other with $\Lambda^* + d\Lambda^*$

Holographic dictionary: Sourced one-point function

$$< \mathcal{O} >^S_{\Lambda^*} \sim \frac{\delta S^{on-shell}_{B,ren.}}{\delta \phi}(r = \Lambda^*) \qquad (17)$$

Green's function $\mathcal{G}_\mathcal{R}$: up to first order in the source

$$< \mathcal{O} >^S_{\Lambda^*} = \lim_{r \to \Lambda^*} \mathcal{G}_R(r)\phi(r) \qquad (18)$$

Computation...

For the theory living at $\Lambda^* + d\Lambda^*$:

$$\mathcal{G}_R(\Lambda^* + d\Lambda^*) \approx \mathcal{G}_R(\Lambda^*) + \frac{1}{\phi(\Lambda^*)} d\Lambda^* \frac{d}{d\Lambda^*} \left(< \mathcal{O} >^S_{\Lambda^*} \right) \quad (19)$$

$$\frac{d\mathcal{G}_R(\Lambda^*)}{d\Lambda^*} = \frac{1}{\phi(\Lambda^*)} \frac{d}{d\Lambda^*} \left(< \mathcal{O} >^S_{r=\Lambda^*} \right) \quad (20)$$

It turns out that $\frac{d}{d\Lambda^*} \left(< \mathcal{O} >^S_{\Lambda^*} \right)$ can be related to the equations of motion of $< \mathcal{O} >^S (r)$ and $\phi(r)$ for the theory defined at Λ^* by formally identifying r with Λ^*!

We can describe the cutoff flow as dynamics in the bulk

Conclusions

- We can define two different theories equipped with a cutoff.
- We have explicitly constructed the flow of $\mathcal{G}_R(\Lambda^*)$ as we vary the position of the cutoff.

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The model

The action writes [Son, Surówka '09]

$$\Delta S = \frac{1}{16\pi G}\int_{r<\Lambda^*}\sqrt{-g}\left(-\frac{1}{4}F_{MN}F^{MN} + \frac{\kappa}{3}\epsilon^{MNPQR}A_M F_{NP}F_{QR}\right)$$

Backgrund metric (RN-AdS BB) and A_μ fields

$$ds^2 = \frac{r^2}{L^2}\left(-f(r)dt^2 + d\bar{x}^2\right) + \frac{L^2}{r^2 f(r)}dr^2 \tag{21}$$

$$f(r) = 1 - \frac{ML^2}{r^4} + \frac{Q^2L^2}{r^6}; \qquad A = \left(\beta - \frac{\mu r_H^2}{r^2}\right)dt \tag{22}$$

Consistent definition of the current

$$J^\mu = \frac{\delta S}{\delta A_\mu(r=\Lambda^*)} = \lim_{r\to\Lambda^*}\frac{\sqrt{-g}}{16\pi G}\left(F^{\mu r} + \frac{4\kappa}{3}\epsilon^{r\mu\nu\rho\lambda}A_\nu F_{\rho\lambda}\right) \tag{23}$$

The model reproduces the correct expression for the axial anomaly of the boundary theory ([Amado, Landsteiner, Pena-Benitez '11])

$$D_\mu J^\mu = \lim_{r\to\Lambda^*}\frac{-\sqrt{-g}\kappa}{48\pi G}\epsilon^{r\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda} \tag{24}$$

Consistent vs. Covariant

- *Consistent definition of the current*
  Response of the system to a perturbation, i.e.

$$< j^\mu > \sim \lim_{r \to \Lambda^*} \frac{\delta S}{\delta A_\mu} \tag{25}$$

- *Covariant definition of the current*
  Redefinition of the current so as to avoid the appearance of non-covariant parts (by adding the BZ polynomial to the consistent current). Holographically: it amounts to taking the subleading term in the asymptotic value of A_μ as definition of the current [Son, Surówka '09].

$$J^\mu = \lim_{r \to \Lambda^*} \frac{\sqrt{-g}}{16\pi G} F^{\mu r} \tag{26}$$

- Surface (such as Chern-Simons) terms flow trivially: We will stick to the covariant definition.

Applying the method to a transport coefficient

Variation of the current due to a normalized external perturbation $a(x,r) = a^{(0)}(x)\frac{a(r)}{a(\Lambda^*)}$:

$$\delta J^\mu = \lim_{r\to\Lambda^*} \frac{\sqrt{-g}}{16\pi G}\delta F^{\mu r} \tag{27}$$

The external source reads $E(\Lambda^*) = -i\omega a^{(0)}(x) = E(\Lambda^* + d\Lambda^*)$. It has the same value for both theories.
Holographic dictionary: The electric conductivity $\sigma_E(\Lambda^*)$ is

$$\sigma_E(\Lambda^*) = \lim_{r\to\Lambda^*} \frac{-1}{16\pi G\omega}\sqrt{-g}g^{xx}g^{rr}\frac{da(r)/dr}{a(\Lambda^*)} \tag{28}$$

Expanding $\sigma_E(\Lambda^* + d\Lambda^*)$ to first order in Λ^* one gets

$$\frac{d\sigma(\Lambda^*)}{d\Lambda^*} = \frac{1}{E^x}\frac{d}{d\Lambda^*}\delta J^x(r=\Lambda^*) \tag{29}$$

Idea: Use bulk equations to describe the cutoff flow!

Computation of the flow for the Electric and Chiral Magnetic Conductivities

Perturbations that we switch on: $a_x(x,r); a_z(x,r)$. In momentum space $a_{(x,z)}(k,r) = a(r)e^{-i\omega t + iky}$ We use *two equations* for the current (the covariant formula and the constitutive one)

$$\delta J^x = \frac{\sqrt{-g}}{16\pi G}\delta F^{\mu r} \tag{30}$$

$$\delta J^x = \sigma_E E^x + \sigma_B B^x \equiv \sigma_E \delta F_{0x} + \sigma_B \epsilon(xjk)\frac{\delta F_{jk}}{2} \tag{31}$$

Equations of motion on Σ_r

$$\frac{1}{\sqrt{-g}}\left[-16\pi G \partial_r j^x + \partial_y(\sqrt{-g}\delta F^{yx}) + \partial_t(\sqrt{-g}\delta F^{tx})\right] = -8\kappa \epsilon^{rtxyz} F_{rt}\delta F_{yz} \tag{32}$$

$$\partial_r j^x + \frac{\sqrt{-g}}{16\pi G}\left(ikg^{yy}g^{xx}B^z + i\omega g^{xx}g^{tt}E^x\right) = -\frac{\kappa}{2\pi G}F_{rt}B^x \tag{33}$$

Bianchi identities: $B^z = \frac{k}{\omega}E^x$

Computation of the flow for the Electric and Chiral Magnetic Conductivities II

Taking the derivative of (31) explicitly

$$\partial_r \delta J^x = \partial_r \sigma_E E^x + \sigma_E \partial_r E^x + \partial_r \sigma_B B^x + \sigma_B \partial_r B^x \quad (34)$$

$$\partial_r B^x = ik\delta F_{rz} = \frac{-ik16\pi G}{\sqrt{-g}} g_{rr} g_{zz} \delta J^z = \frac{-ik16\pi G}{\sqrt{-g}} g_{rr} g_{zz} \left[\sigma_E E^z + \sigma_B B^z\right] \quad (35)$$

Using the Bianchi identities again...

$$\partial_r B^x = \frac{-ik16\pi G}{\sqrt{-g}} g_{rr} g_{zz} \left[\sigma_E \frac{-\omega}{k} B^x + \sigma_B \frac{k}{\omega} E^x\right] \quad (36)$$

Finally, one gets

$$\partial_r j^x = E^x \left[\partial_r \sigma_E + \frac{i\omega 16\pi G}{\sqrt{-g}} g_{rr} g_{xx} \left(\sigma_E^2 - \frac{k^2}{\omega^2}\sigma_B^2\right)\right] + \\ + B^x \left[\partial_r \sigma_B + \frac{i\omega 32\pi G}{\sqrt{-g}} g_{rr} g_{xx} \sigma_B \sigma_E\right] \quad (37)$$

Computation of the flow for the Electric and Chiral Magnetic Conductivities III

Equating (33) and (37) we arrive at

$$\partial_r j^x = E^x \left[\partial_r \sigma_E + \frac{i\omega 16\pi G}{\sqrt{-g}} g_{rr} g_{xx} \left(\sigma_E^2 - \frac{k^2}{\omega^2}\sigma_B^2\right)\right] + \\ + B^x \left[\partial_r \sigma_B + \frac{i\omega 32\pi G}{\sqrt{-g}} g_{rr} g_{xx} \sigma_B \sigma_E\right] = \\ = -\frac{\sqrt{-g}}{16\pi G}\left(i\frac{k^2}{\omega} g^{yy} g^{xx} E^x + i\omega g^{xx} g^{tt} E^x\right) - \frac{\kappa}{2\pi G} F_{rt} B^x \tag{38}$$

And, finally

$$\partial_r \sigma_E = -i\omega \left[\frac{16\pi G}{\sqrt{-g}} g_{rr} g_{xx} \left(\sigma_E^2 - \frac{k^2}{\omega^2}\sigma_B^2\right) + \frac{\sqrt{-g}}{16\pi G} g^{xx}\left(g^{tt} + \frac{k^2}{\omega^2} g^{yy}\right)\right] \tag{39}$$

$$\partial_r \sigma_B = -i\omega \frac{32\pi G}{\sqrt{-g}} g_{rr} g_{xx} \sigma_B \sigma_E - \frac{\kappa}{2\pi G} F_{rt} \tag{40}$$

Computation of the flow for the Electric and Chiral Magnetic Conductivities IV

Regularity at the horizon $r = r_H$ imposes:

$$\partial_r A_i = \sqrt{\frac{g_{rr}}{-g_{tt}}} \partial_t A_i \tag{41}$$

This fact can be seen to be a consequence of infalling boundary conditions (*membrane pàradigm*). In our gauge choice $A_r = 0$, the above relation implies

$$F_{ri} = \sqrt{\frac{g_{rr}}{-g_{tt}}} F_{ti} \tag{42}$$

The horizon is compatible only with an electric conductivity σ_E. In particular, $\sigma_B(r = r_H) = 0$

In the limit $\omega, k \to 0$, (39) and (40) reduce to

$$\partial_r \sigma_E = 0 \tag{43}$$

$$\partial_r \sigma_B = -\frac{\kappa}{2\pi G} F_{rt} = \frac{\mu r_H^2}{2\pi^2 r^3} \tag{44}$$

Computation of the flow for the Electric and Chiral Magnetic Conductivities V

Solutions to the first order equations

$$\sigma_E = Constant \tag{45}$$

$$\sigma_B(r) = \frac{\mu}{4\pi^2}\left(1 - \frac{r_H^2}{r^2}\right) \tag{46}$$

- Equation (45) has appeared before in [Iqbal, Liu '08], implying the universality of σ_E.
- σ_B reduces to $\frac{\mu}{4\pi^2}$ in the limit $r \to \infty$, as expected from [Amado, Landsteiner, Pena-Benitez '11]. Notice that we are able to take naively the limit $r \to \infty$ for $S_A^{cterm} \sim \int d^4xF^2$ is of order k^2.

Result and emergence of $\mu(\Lambda)$

The energy-momentum tensor reads

$$t^a_b = \lim_{r\to\Lambda^*} \frac{\sqrt{-\gamma}}{8\pi G}(\delta^a_b K - K^a_b) \tag{47}$$

(γ is the determinant of the induced metric and K^a_b is the extrinsic curvature).
Repeating the process...

$$\sigma_V(r) = \frac{\mu^2}{8\pi^2}\left(1 - \frac{2r_H^2}{r^2} + \frac{r_H^4}{r^4}\right) \tag{48}$$

Emerging chemical potential

$$\mu(r) = \mu\left(1 - \frac{r_H^2}{r^2}\right) \tag{49}$$

Preliminary calculation of σ^ϵ_V, σ^ϵ_B points towards this effective $\mu(r)$ as well

Non-renormalization theorem

Non-Renormalization Theorem: The weak coupling limit captures the full dynamics of the anomalies. Therefore, anomalous transport coefficients computed at weak and strong coupling coincide.

	Weak Coupling	Strong Coupling
σ_B	$\frac{\mu}{4\pi^2}$ [1]	$\frac{\mu}{4\pi^2}$ [2]
σ_V	$\frac{\mu^2}{8\pi^2}$ [3]	$\frac{\mu^2}{8\pi^2}$ [4]
σ_B^ϵ	$\frac{\mu^2}{8\pi^2}$ [5]	$\frac{\mu^2}{8\pi^2}$
σ_V^ϵ	$\frac{\mu^3}{12\pi^2}$ [6]	$\frac{\mu^3}{12\pi^2}$

[1] [Kharzeev, Warringa '09]; [Newman '07]
[2] [Son, Surówka '09]
[3] [Landsteiner, Megías, Pena-Benitez '11]
[4] [Erdmenger, Haak, Kaminski,Yarom '09]; [Banerjee, Bhattacharya, Bhattacharya, Dutta, Loganayagam]
[5] "Strongly interacting matter in a magnetic field". Springer
[6] "Strongly interacting matter in a magnetic field". Springer

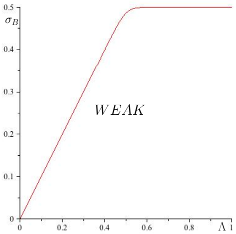
Explicit function $\Lambda(r)$

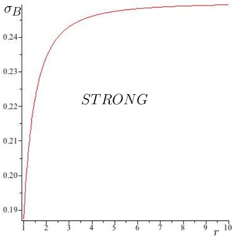
Weak coupling: QFT approach [Kharzeev, Warringa '09]

$$\sigma_B(\Lambda) = \int_0^\Lambda dq\,(n(E_q-\mu) - n(E_q+\mu)) = \mathcal{F}(\Lambda) - \mathcal{F}(0)$$

Strong coupling: AdS/CFT $\rightarrow \sigma_B(r) = \frac{\mu^2}{4\pi^2}\left(1 - \frac{r_H^2}{r^2}\right)$ *Non-renormalization theorem* $\rightarrow \Lambda(r) = \mathcal{F}^{-1}(\sigma_B(r) + \mathcal{F}(0))$

Explicit map between Λ *and* r





Interpretation

- *Effective* $\mu(\Lambda)$
  The chemical potential represents the necesary energy to introduce a unit of charge into the system. It depends on the cutoff scale: The unit of charge must have a wavelenght of, at most, $\Delta\lambda \sim \frac{1}{\Lambda}$. Therefore, it demands less energy to enter the system and spread into it (thermalize).

- $\dot{\sigma}_E = 0$
  The electric conductivity is purely dominated by IR contributions. This fact hints at the reason why it is so difficult to compute these conductivities in non-abelian gauge theories such as QCD. One could expect that $\sigma_E = constant$ is an $N_c \to \infty$ effect, and that $\frac{1}{N_c}$ contributions would make $\dot{\sigma}_E(r)$ become a varying function that approaches 0 as $r - r_H$ increases.

Conclusions and perspectives

- We have studied the cutoff flow of different transport coefficients, showing that, surprisingly, the anomalous ones depend explicitly on the cutoff. With this fact in mind, we have analyzed the possibility of finding an explicit map $\Lambda(r)$, providing that the Non-renormalization theorem applies.

- The flow can be interpreted as an effective flow of the chemical potential. Preliminary calculations of the remaining conductivities support this point.

- Remaining issues
  - Compute the flow for $\sigma_\epsilon^B, \sigma_\epsilon^V$.
  - Include the gravitational anomaly into the computation. [Landsteiner, Megías, Melgar, Pena-Benitez '11]
  - Generalize the background metric.
  - Compute the proper Wilsonian RG flow and study the behaviour of the transport coefficients. [Faulkner, Liu, Rangamani '11]